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Journal of Geometry and Physics 56 (2006) 2093–2108
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Classification of the Weyl curvature spinors of neutral metrics in four dimensions

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Received 5 July 2005; received in revised form 8 November 2005; accepted 10 November 2005
Available online 27 December 2005

Abstract

Neutral geometry is of increasing interest. As with Riemannian and Lorentzian geometry, spinors can be expected to provide a valuable tool in neutral geometry. For a neutral metric in four dimensions, the classification of the Weyl curvature spinors by the pattern of principal spinors each admits is given. For each Weyl curvature spinor, there are nine nontrivial types. This classification is then related to the classification, given previously by the author, of a Weyl curvature spinor when regarded as a curvature endomorphism (four types). These results are the neutral analogues of well known and fundamental results in Lorentzian geometry, but display the peculiarities of neutral geometry. One can expect these results to be an essential ingredient in a full understanding of neutral geometry in four dimensions.

MSC: 53B30; 15A66; 83C60

JGP: Real and complex differential geometry; Spinors and twistors

Keywords: Neutral geometry; Weyl curvature; Spinors; Curvature classification

1. Introduction and notation

There is growing interest in neutral geometry, i.e., the geometry of pseudo-Riemannian manifolds equipped with a metric of signature (n, n). Matsushita [29–31], and reviewed in [33], studied existence conditions for neutral metrics in four dimensions admitting a reduction of the orthogonal group $\mathbf{O(2,2)}$ to its identity-connected component $\mathbf{SO^+(2,2)}$; in particular,

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doi:10.1016/j.geomphys.2005.11.008

such a reduction entails the existence of a pair of almost complex structures which induce opposite orientations and thus falls naturally into the (almost) complex category. Law [22,23] emphasized the role of the neutral orthogonal group **NO(n)** (of all orthogonal and anti-orthogonal transformations) in neutral geometry. Matsushita and Law have studied the classification of curvature for neutral metrics in four dimensions and the analogue of the Thorpe–Hitchin inequality for neutral Einstein metrics on compact four-manifolds [21,35].

One of the intriguing aspects of neutral geometry is the occurrence of parallels with Riemannian geometry in the context of indefinite signature, already evident in [21]. Others have studied various neutral analogues of features of Riemannian geometry: parallel mean curvature surfaces [14]; self-dual metrics in four dimensions [19,15,6,4]; Kähler geometry [39,43,17,18, 7]; hyperKähler geometry [13,16,9,7].

Standard geometrical topics have also attracted attention, e.g., holonomy [2,10] and Jacobi operators [3,8]. Recently, Matsushita and coworkers [32,5] began a study of neutral four-manifolds admitting a field of parallel totally null planes, exploiting the canonical form of the metric in such circumstances provided by Walker [44]. This condition is very natural in neutral geometry and has already yielded the result that the almost complex structure of a compact almost Kähler–Einstein neutral manifold need not be integrable, i.e., 'Goldberg's conjecture' [11] fails for neutral signature, see [34].

There is also a real version of twistor theory for neutral signature in four dimensions which has been investigated sporadically, for example [12,47,6,26], but also plays a role in integrable systems, for example via the Jones–Tod correspondence which yields three-dimensional Lorentzian Einstein–Weyl geometries by a symmetry reduction on appropriate neutral four-manifolds, see [28,6], but also [27].

Neutral geometry is also the real geometry underlying smooth and holomorphic complex Riemannian geometry, see [46,42,41,24].

Possible applications of neutral geometry in physics have stimulated some of the above work, and more [36]; see [45] for a recent example of the use of $(+ + --)$ signature in string theory. This brief literature review, which is not intended to be exhaustive, indicates an interest in neutral geometry that is, I believe, well deserved. Familiar features of Riemannian geometry, subtly warped by neutral signature, illuminate both Riemannian and neutral geometry.

Spinors have proven a powerful tool in both Riemannian [25] and Lorentzian geometry [37, 38] and have already attracted interest in neutral geometry [47,6]. In this paper I classify the algebraic structure of the Weyl curvature spinor of a neutral metric in four dimensions and relate this classification to that of the Weyl curvature endomorphisms in [21], thereby providing the neutral analogue of well known results in general relativity [38]. This particular feature of neutral geometry has, of course, no parallel in Riemannian geometry.

I denote by $\mathbf{R}^{p,q}$ the pseudo-Euclidean space consisting of $\mathbf{R}^{p+q}$ equipped with the scalar product whose components with respect to the standard basis form the diagonal matrix, the first p diagonal entries of which are all 1, the remaining q being -1. The Clifford algebra of $\mathbf{R}^{p,q}$ I denote by $\mathbf{R}_{p,q}$. Similarly, $\mathbf{C}_n$ denotes the Clifford algebra of $\mathbf{C}^n$ equipped with the standard $\mathbf{C}$-bilinear dot product. By $\mathbf{K}(n)$ I denote the algebra of $n \times n$ matrices with entries in $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$). For any $\mathbf{K}$-linear space V, $V_\bullet$ shall denote the dual space; and for any homomorphism T, $T_\bullet$ the dual mapping. When V is finite dimensional and equipped with a ($\mathbf{K}$-bilinear) scalar product s, the scalar product provides an isomorphism between V and $V_\bullet$ which intertwines between the natural actions of the 'orthogonal' group of (V, s) on V and $V_\bullet$. In this context, as usual, one may use the covariant or contravariant form of an abstract index as convenience dictates.

2. Spinor algebra

As is well known, see, e.g., [40,38,25], $\mathbf{C}_{2k} \cong \mathbf{C}(2^k)$ as algebras, with the even part of the algebra $\mathbf{C}^0_{2k} \cong \mathbf{C}_{2k-1} \cong \begin{pmatrix} \mathbf{C}(2^{k-1}) & 0 \\ 0 & \mathbf{C}(2^{k-1}) \end{pmatrix}$, which entails that **Spin(2k;C)** acts reducibly on $\mathbf{C}^{2k-1} \oplus \mathbf{C}^{2k-1}$. Writing these summands as $\mathcal{S}$ and $\mathcal{S}'$, then $\mathbf{C}_{2k} \cong \mathrm{End}_{\mathbf{C}}(\mathcal{S} \oplus \mathcal{S}')$. Moreover $\mathbf{C}^{2k}$ itself has a copy lying in the odd part of $\mathbf{C}_{2k}$: $\begin{pmatrix} 0 & \mathrm{Hom}(\mathcal{S}', \mathcal{S}) \\ \mathrm{Hom}(\mathcal{S}, \mathcal{S}') & 0 \end{pmatrix}$. Now, the equation $\dim_{\mathbf{C}}(\mathrm{Hom}(\mathcal{S}', \mathcal{S})) = 2^{2k-2} = 2k = \dim_{\mathbf{C}}(\mathbf{C}^{2k})$ has the unique integral solution $k = 2$, i.e., for $k = 2$ only does one obtain $\mathbf{C}^{2k} \cong \mathrm{Hom}(\mathcal{S}', \mathcal{S})$ thereby permitting an identification of $V = \mathbf{C}^4$ with $\mathcal{S} \otimes \mathcal{S}'_\bullet$. This observation is the basis of the complex two-component spinor formalism [37, 38] and its real forms.

In the case of interest here, $\mathbf{R}_{2,2} \cong \mathbf{R}(4)$ and one can obtain a concrete representation as follows. Putting

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

then A, B and C anticommute with each other and $ABC = 1_2$. Now put

$$E_1 := \begin{pmatrix} 0_2 & A \\ A & 0_2 \end{pmatrix} \quad E_2 := \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \quad E_3 := \begin{pmatrix} 0_2 & C \\ C & 0_2 \end{pmatrix} \quad E_4 := \begin{pmatrix} 0_2 & B \\ B & 0_2 \end{pmatrix}. \tag{2}$$

The E_i anticommute with each other, and satisfy $-1_2 = (E_1)^2 = (E_2)^2 = -(E_3)^2 = -(E_4)^2$ and $\Lambda := E_1E_2E_3E_4 = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}$. Thus, $\{E_1, E_2, E_3, E_4\}$ generate $\mathbf{R}_{2,2}$ as an algebra and serve as a pseudo-orthonormal (Ψ-ON) basis for a copy of $\mathbf{R}^{2,2}$ within $\mathbf{R}_{2,2}$. Explicitly, $(u, v, x, y) \in \mathbf{R}^{2,2}$ is identified with

$$uE_1 + vE_2 + xE_3 + yE_4 = \begin{pmatrix} 0_2 & Z \\ -{}^*Z & 0_2 \end{pmatrix} \qquad Z := \begin{pmatrix} v - y & x + u \\ x - u & v + y \end{pmatrix}, \tag{3}$$

where ${}^*Z$ denotes the adjoint of Z wrt the symplectic plane $\mathbf{R}^2_{\mathrm{sp}}$ (in matrix form taken with respect to the standard basis).

Now Λ splits the pinor space into the direct sum decomposition $S \oplus S'$, each summand a copy of $\mathbf{R}^2$, and thereby introduces a 2×2 blocking of elements of $\mathbf{R}(4)$. As usual, use unprimed indices for elements of S and primed indices for elements of S', but note that these two spaces are independent; there is no mapping from one to the other. By a general property of Clifford algebras, Λ commutes with the even subalgebra $\mathbf{R}^0_{2,2}$ and anticommutes with the odd part, whence $\mathbf{R}^0_{2,2} \cong \begin{pmatrix} \mathbf{R}(2) & 0_2 \\ 0_2 & \mathbf{R}(2) \end{pmatrix}$. Clifford conjugation, i.e., the algebra anti-involution induced by the negative of the identity transformation of the copy of $\mathbf{R}^{2,2}$ in $\mathbf{R}_{2,2}$ is given by, with $\alpha, \beta, \gamma, \delta \in \mathbf{R}(2)$,

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}^- := \begin{pmatrix} {}^*\alpha & {}^*\beta \\ {}^*\gamma & {}^*\delta \end{pmatrix}, \tag{4}$$

where ${}^*\alpha$, etc., are again the adjoints with respect to $\mathbf{R}^2_{\mathrm{sp}}$. It follows that

$$\mathbf{Spin}^+(\mathbf{2}, \mathbf{2}) \cong \begin{pmatrix} \mathbf{SL}(\mathbf{2}; \mathbf{R}) & 0_2 \\ 0_2 & \mathbf{SL}(\mathbf{2}; \mathbf{R}) \end{pmatrix}, \tag{5}$$

and each spin space is to be identified in fact with a copy of $\mathbf{R}^2_{\text{sp}}$. The symplectic forms will be denoted by ϵ and ϵ' (in the latter case, when indices are employed, the prime will be attached to the indices and not the ϵ). Moreover, the vector representation of $\mathbf{Spin}^+(\mathbf{2},\mathbf{2})$ is given by

$$\begin{pmatrix} \alpha & 0_2 \\ 0_2 & \beta \end{pmatrix} \begin{pmatrix} 0_2 & Z \\ -{}^*Z & 0_2 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0_2 \\ 0_2 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 0_2 & \alpha Z\beta^{-1} \\ -{}^*(\alpha Z \beta^{-1}) & 0_2 \end{pmatrix},$$

exploiting $\mathbf{SL(2;R)} = \mathbf{Sp(2;R)}$.

It suffices to consider $(\alpha, \beta).Z = \alpha Z \beta^{-1}$ for the vector representation. Now $Z \in \mathrm{Hom}(S', S) \cong S \otimes S'_\bullet$. One can therefore use the correlation $\epsilon' : S' \to S'_\bullet$ (actually its inverse) to obtain an identification of $\mathbf{R}^{2,2}$ with $S \otimes S'$. Putting $X := Z(\epsilon')^{-1}$, then, as $\beta \in \mathbf{SL(2;R)} = \mathbf{Sp(2;R)}$, $\alpha Z\beta^{-1}(\epsilon')^{-1} = \alpha Z(\epsilon')^{-1}\epsilon'\beta^{-1}(\epsilon')^{-1} = \alpha X \epsilon' \beta^{-1}_{\bullet\bullet}(\epsilon')^{-1} = \alpha X\, {}^*(\beta^{-1}_\bullet) = \alpha X \beta_\bullet = \alpha\beta X$. Concretely,

$$X^{AA'} := Z^A{}_{B'}\epsilon^{A'B'} = \begin{pmatrix} u+x & y-v \\ y+v & u-x \end{pmatrix}, \tag{6}$$

and the vector representation takes the matrix equation form

$$(\alpha, \beta).X = \alpha.X.{}^\tau\beta, \tag{7}$$

where ${}^\tau\beta$ denotes the transpose of β. This form is compatible with the familiar two-component complex spinor formalism [37,38] (though it is customary to insert a factor of $1/\sqrt{2}$ on the far right hand side of (6)); the natural action of $\mathbf{SL(2;R)}$ on $\mathcal{S} \otimes \mathcal{S}'$ induced from the action on each spin space thus translates directly into the vector representation of $\mathbf{Spin}^+(\mathbf{2},\mathbf{2})$ on $\mathbf{R}^{2,2}$ via the identification with $\mathcal{S} \otimes \mathcal{S}'$ provided by X, with the quadratic form given by a suitable multiple of the determinant of X. In particular, when $X = \xi \otimes \eta$, $\xi \in \mathcal{S}$, $\eta \in \mathcal{S}'$, then (7) is $X \mapsto (\alpha.\xi) \otimes (\beta.\eta)$.

Much of the familiar two-component spinor formalism of [37] carries over for $\mathbf{R}^{2,2}$, adjusting for the reality of the spin spaces. For example, the null cone K of the origin in $\mathbf{R}^{2,2}$ is foliated (strictly speaking, when the vertex is excluded) by each of two families of totally null planes, each family parametrized by $\mathbf{SO(2)}$. The members of one of these families (α-planes) may be described as $Z_\theta := \{(z, R_\theta(z)) \in \mathbf{R}^{2,2} : z \in \mathbf{R}^2\}$, $R_\theta \in \mathbf{SO(2)}$ the standard rotation through θ, which under (6), and writing $z = (u, v)$, is

$$Z_\theta = \left\{ \eta^A \pi^{A'} : \pi^{A'} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}^{A'}, \eta^A = u \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}^{A} + v \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}^{A} \right\}.$$

Note that Z_θ is characterized by the projective class of $\pi^{A'}$, so one could equally well write $Z_{[\pi]}$, while η^A varies over all of S, and that as $\theta : 0 \to 2\pi$, $\pi^{A'} : \binom{1}{0} \to \binom{-1}{0}$, manifesting the two-valuedness of spinors.

The elements of the other family (β-planes) may be written as $W_\theta = \{(w, L_\theta(w)) \in \mathbf{R}^{2,2} : w \in \mathbf{R}^2\}$, where $L_\theta = TR_\theta$, $T \in \mathbf{O(2)} \setminus \mathbf{SO(2)}$. With $w = (u, v)$ and $T(u, v) := (-u, v)$, under (6),

$$W_\theta = \left\{ \pi^A \eta^{A'} : \pi^{A} = \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}^{A}, \eta^{A'} = u \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}^{A'} + v \begin{pmatrix} \cos(\theta/2) \\ -\sin(\theta/2) \end{pmatrix}^{A'} \right\}.$$

Hence, there is an $\mathbf{S}^1$'s worth of α-planes lying on K, and each contains an $\mathbf{S}^1$'s worth of null directions through the origin. Any two of these α-planes intersect only at the origin, and the null directions fill out K, i.e., as is well known, the quadric Grassmannian of lines on K is $\mathbf{S}^1 \times \mathbf{S}^1$. The same is true of the β-planes lying on K. Any one of these α-planes and any one of the β-planes has a null direction through the origin as their intersection. This structure is repeated for each null cone with vertex an arbitrary point of $\mathbf{R}^{2,2}$, giving rise to the families of affine α- and β-planes.

As noted, $Z_\theta = Z_{[\pi]}$ determines $\pi^{A'}$ only up to scale. The anti-self-dual (ASD) two-form $F_{ab} := \epsilon_{AB}\pi_{A'}\pi_{B'}$, which as an element of $\mathrm{Hom}(\mathbf{R}^4, (\mathbf{R}^4)_\bullet)$ has $Z_{[\pi]}$ as kernel, determines $\pi^{A'}$ up to sign.

It will be convenient to write, in general, $\epsilon(\alpha^A, \beta^B) = \epsilon_{AB}\alpha^A\beta^B =: \beta \cdot \alpha$.

Since my aim in this paper is the classification of the Weyl (curvature) spinor, consider totally symmetric spinors: $\Phi^{AB\ldots M} = \Phi^{(AB\ldots M)}$. As usual, by virtue of the fundamental theorem of algebra, one can write

$$\Phi^{AB\ldots M} = \alpha^{(A}\beta^B \ldots \mu^{M)}, \tag{8}$$

where one must allow for the possibility of complex spinors in this representation, i.e., elements of $\mathbf{C}\mathcal{S}$, which of course will occur in complex conjugate pairs. If $\alpha^A \in \mathbf{C}\mathcal{S}$, write $\alpha^A = \gamma^A + i\delta^A$, $\gamma^A, \delta^A \in \mathcal{S}$. Then $\overline{\alpha} \cdot \alpha = 2i(\gamma \cdot \delta)$. Since $\gamma \cdot \delta = 0$ is equivalent to $\alpha^A = z\gamma^A$, for some $z \in \mathbf{C}$, i.e., α^A is just a complex multiple of a real spinor, I shall assume that any complex spinor α^A appearing in a decomposition such as (8) satisfies

$$\overline{\alpha} \cdot \alpha \neq 0. \tag{9}$$

The solutions of (8) will be called *principal spinors* of $\Phi^{AB\ldots M}$, which are determined only up to scale of course. Unlike in the Lorentzian case, however, there is no notion of principal null direction. A totally symmetric rank two spinor $\Phi^{AB} = \alpha^{(A}\beta^{B)}$ is called *null* iff $\beta \cdot \alpha = 0$. A null Φ^{AB} has a real principal spinor. As $\Phi^{AB}\Phi_{AB} = -(\beta \cdot \alpha)^2/2$,

$$\text{nullity} \Leftrightarrow \Phi^{AB}\Phi_{AB} = 0. \tag{10}$$

My considerations in this paper will be entirely local; but it is clear that to employ the notion of spinors transforming under $\mathbf{Spin}^+(\mathbf{2}, \mathbf{2})$, i.e., a *spinor structure*, necessitates a reduction of $\mathbf{O}(\mathbf{2}, \mathbf{2})$ to $\mathbf{SO}^+(\mathbf{2}, \mathbf{2})$ in the relevant circumstances. For global applications, one requires such a reduction to $\mathbf{SO}^+(\mathbf{2}, \mathbf{2})$ for the frame bundle of the manifold, in which case the remaining obstruction to a spinor structure on the manifold is the nonvanishing of the second Stiefel–Whitney class of the tangent bundle [20,1].

3. The Weyl curvature spinors

The spinor form of the curvature tensor of a neutral metric in four dimensions is derived formally exactly as in [37], except that the various curvature spinors are all real. In particular, the Weyl conformal curvature takes the form $C^{abcd} = \Psi^{ABCD}\epsilon^{A'B'}\epsilon^{C'D'} + \Psi^{A'B'C'D'}\epsilon^{AB}\epsilon^{CD}$, where Ψ^{ABCD} and $\Psi^{A'B'C'D'}$ are independent, real totally symmetric spinors belonging to $\mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}$ and $\mathcal{S}' \otimes \mathcal{S}' \otimes \mathcal{S}' \otimes \mathcal{S}'$ respectively, called the *Weyl curvature spinors*. In particular,

$$^{-}C^{abcd} := \Psi^{ABCD}\epsilon^{A'B'}\epsilon^{C'D'} \qquad ^{+}C^{abcd} := \Psi^{A'B'C'D'}\epsilon^{AB}\epsilon^{CD}, \tag{11}$$

are called the ASD and SD parts respectively. It will suffice to consider the ASD part and the corresponding Weyl spinor Ψ^{ABCD}. By (8), one can write

$$\Psi^{ABCD} = \alpha^{(A}\beta^{B}\gamma^{C}\delta^{D)}, \tag{12}$$

where α^A, β^B, γ^C, δ^D will be called *Weyl principal spinors*, abbreviated as WPS.

For arbitrary spinors $\xi^A \in \mathcal{S}$, $\eta^{A'} \in \mathcal{S}'$, put $v^a = \xi^A\eta^{A'}$, a null vector. By a straightforward adaptation of the argument in [38, p. 224], one finds: ξ^A is a WPS of a nonzero Ψ^{ABCD} iff

$$\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0, \qquad \text{equivalently } v_{[f}\,{}^{-}C_{a]bc[d}v_{e]}\,v^b v^c = 0; \tag{13a}$$

of multiplicity at least two iff

$$\Psi_{ABCD}\xi^A\xi^B\xi^C = 0, \qquad \text{equivalently } {}^{-}C_{abc[d}\,v_{e]}v^b v^c = 0; \tag{13b}$$

of multiplicity at least three iff

$$\Psi_{ABCD}\xi^A\xi^B = 0, \qquad \text{equivalently } {}^{-}C_{abc[d}\,v_{e]}v^c = 0; \tag{13c}$$

and of multiplicity four iff

$$\Psi_{ABCD}\xi^A = 0, \qquad \text{equivalently } {}^{-}C_{abcd}\,v^c = 0. \tag{13d}$$

There are obvious analogous conditions on ${}^{+}C^{abcd}$ for $\eta^{A'}$ to be a WPS of a nonzero $\Psi^{A'B'C'D'}$.

The possible distinct *types* of Ψ^{ABCD} are described and coded as follows. With α, β, γ, and δ distinct elements of $\mathbf{P}\mathcal{S}$,

$$\begin{aligned} &\{1111\} : \Psi^{ABCD} = \alpha^{(A}\beta^{B}\gamma^{C}\delta^{D)} \qquad \{211\} : \Psi^{ABCD} = \alpha^{(A}\alpha^{B}\gamma^{C}\delta^{D)} \\ &\{22\} : \Psi^{ABCD} = \alpha^{(A}\alpha^{B}\delta^{C}\delta^{D)} \qquad \{31\} : \Psi^{ABCD} = \alpha^{(A}\alpha^{B}\alpha^{C}\delta^{D)} \\ &\{4\} : \Psi^{ABCD} = \alpha^{A}\alpha^{B}\alpha^{C}\alpha^{D}. \end{aligned} \tag{14a}$$

With α, β distinct elements of $\mathbf{P}(\mathbf{C}\mathcal{S})$ and γ, δ distinct elements of $\mathbf{P}\mathcal{S}$,

$$\begin{aligned} &\{1\bar{1}11\} : \Psi^{ABCD} = \alpha^{(A}\overline{\alpha}^{B}\gamma^{C}\delta^{D)} \qquad \{1\bar{1}1\bar{1}\} : \Psi^{ABCD} = \alpha^{(A}\overline{\alpha}^{B}\beta^{C}\overline{\beta}^{D)} \\ &\{1\bar{1}2\} : \Psi^{ABCD} = \alpha^{(A}\overline{\alpha}^{B}\delta^{C}\delta^{D)} \qquad \{2\bar{2}\} : \Psi^{ABCD} = \alpha^{(A}\overline{\alpha}^{B}\alpha^{C}\overline{\alpha}^{D)}. \end{aligned} \tag{14b}$$

Together with $\{-\}$ to indicate vanishing Ψ^{ABCD}, there are 10 types for Ψ^{ABCD}. There are three *algebraically general* types: $\{1111\}$, $\{1\bar{1}11\}$, and $\{1\bar{1}1\bar{1}\}$. All other cases are *degenerate* forms of these three and are said to be *algebraically special*.

4. The Weyl curvature endomorphism

In this section I relate the endomorphism ${}^{-}C^{ab}{}_{cd}$ of the space of ASD two-multivectors of $\mathbf{R}^{2,2}$ to the endomorphism $\Psi^{AB}{}_{CD}$ of $\mathcal{S}^{(AB)}$, the space of totally symmetric elements of $\mathcal{S} \otimes \mathcal{S}$. Let Λ^2 denote the space of two-multivectors of $\mathbf{R}^{2,2}$. The Hodge $*$ operator is an involution of this six-dimensional space; its eigenspaces are, by definition, the spaces of SD and ASD two-multivectors. This structure pertains to the tangent space at each point of a four-manifold M equipped with a neutral metric g, indeed with a neutral structure as defined in [23]. In accordance with the remark at the end of the introduction, I describe the situation in terms of multivectors, rather than two-forms.

The spinor form of elements of Λ^2_-, the space of ASD two-multivectors, is $\Phi^{AB}\epsilon^{A'B'}$, where $\phi^{AB} \in \mathcal{S}^{(AB)}$. Hence $\Lambda^2_- \cong \mathcal{S}^{(AB)}$ and the action of ${}^-C^{ab}{}_{cd}$ on Λ^2_- is given by that of $\Psi^{AB}{}_{CD}$ on $\mathcal{S}^{(AB)}$. Explicitly, with $F^{ab} = \phi^{AB}\epsilon^{A'B'}$ and $H^{ab} = \psi^{AB}\epsilon^{A'B'}$in Λ^2_-, take the induced scalar product on Λ^2_- to be

$$s(F, H) = \frac{1}{2}F^{ab}H_{ab} = \frac{1}{2}\phi^{AB}\psi_{AB}\epsilon^{A'B'}\epsilon_{A'B'} = \phi^{AB}\psi_{AB}. \tag{15}$$

By (10), the null elements with respect to this scalar product are precisely the *null* elements in the sense of Section 2. Since $\Lambda^2_- \cong \mathcal{S}^{(AB)}$, we expect $(\mathcal{S}^{(AB)}, s) \cong \mathbf{R}^{1,2}$. Indeed, with respect to a basis, the elements of $\mathcal{S}^{(AB)}$ are represented by symmetric elements of $\mathbf{R}(2)$ (the space of which I denote $\mathbf{R}_+(2)$) with $\phi^{AB}\phi_{AB} = 2\det(\phi^{\mathbf{AB}})$. The subspace of diagonal symmetric matrices is therefore isomorphic to $\mathbf{R}^{1,1}$ while the subspace of symmetric matrices with zero diagonal is isomorphic to $\mathbf{R}^{0,1}$. The action of $\mathbf{SL(2;R)}$ on $\mathcal{S}^{(AB)}$ is given by $\phi^{AB} \mapsto \alpha^A{}_C\alpha^B{}_D\phi^{CD}$, for $\alpha \in \mathbf{SL(2;R)}$, which with respect to a basis takes the matrix equation form

$$\phi \mapsto \alpha.\phi.{}^\tau\alpha\,. \tag{16}$$

One way to see the significance of this action is as follows. The Clifford algebra $\mathbf{R}_{1,2}$ is isomorphic to $\begin{pmatrix}\mathbf{R}(2) & 0_2\\ 0_2 & \mathbf{R}(2)\end{pmatrix}$. Indeed, with A, B and C as in (1) then $F_1 := \begin{pmatrix}A & 0_2\\ 0_2 & -A\end{pmatrix}$, $F_2 := \begin{pmatrix}B & 0_2\\ 0_2 & -B\end{pmatrix}$ and $F_3 := \begin{pmatrix}C & 0_2\\ 0_2 & -C\end{pmatrix}$ generate $\mathbf{R}_{1,2}$ as an algebra and serve as a Ψ-ON basis for a copy of $\mathbf{R}^{1,2}$ within $\mathbf{R}_{1,2}$. So, $te_1 + xe_2 + ye_3 \in \mathbf{R}^{1,2}$ is identified with $\begin{pmatrix}Z & 0_2\\ 0_2 & -Z\end{pmatrix}$, where $Z = \begin{pmatrix}-x & t+y\\ y-t & x\end{pmatrix}$. With $\alpha, \beta \in \mathbf{R}(2)$, the main involution of $\mathbf{R}_{1,2}$, i.e., the involution induced by the negative identity transformation of the copy of $\mathbf{R}^{1,2}$ in $\mathbf{R}_{1,2}$, is given by $\begin{pmatrix}\alpha & 0_2\\ 0_2 & \beta\end{pmatrix}^\dagger = \begin{pmatrix}\beta & 0_2\\ 0_2 & \alpha\end{pmatrix}$ while Clifford conjugation is given by $\begin{pmatrix}\alpha & 0_2\\ 0_2 & \beta\end{pmatrix}^- = \begin{pmatrix}{}^*\alpha & 0_2\\ 0_2 & {}^*\beta\end{pmatrix}$, where $*$ denotes the taking of adjoints with respect to $\mathbf{R}^2_{\mathrm{sp}}$. It follows that $\mathbf{R}^0_{1,2} \cong \left\{\begin{pmatrix}\alpha & 0_2\\ 0_2 & \alpha\end{pmatrix} : \alpha \in \mathbf{R}(2)\right\}$ and $\mathbf{Spin}^+(\mathbf{1},\mathbf{2}) \cong \left\{\begin{pmatrix}\alpha & 0_2\\ 0_2 & \alpha\end{pmatrix} : \alpha \in \mathbf{SL(2;R)}\right\}$. The vector representation of $\mathbf{Spin}^+(\mathbf{1},\mathbf{2})$ is

$$\begin{pmatrix}\alpha & 0_2\\ 0_2 & \alpha\end{pmatrix}\begin{pmatrix}Z & 0_2\\ 0_2 & -Z\end{pmatrix}\begin{pmatrix}\alpha & 0_2\\ 0_2 & \alpha\end{pmatrix}^{-1} = \begin{pmatrix}\alpha Z\alpha^{-1} & 0_2\\ 0_2 & -\alpha Z\alpha^{-1}\end{pmatrix}.$$

It therefore suffices to consider the action as given by $\alpha.Z = \alpha Z\alpha^{-1}$. Put

$$X^{AB} := Z^A{}_C\epsilon^{BC} = \begin{pmatrix}t+y & x\\ x & t-y\end{pmatrix} = t1_2 + y(-B) + xC \in \mathbf{R}_+(2). \tag{17}$$

By the same argument as leads to (7), the vector representation may be written $\alpha.X = \alpha.X.{}^\tau\alpha$, i.e., (16). Thus, under an $\mathbf{SL(2;R)}$ transformation of $\mathcal{S}$, the induced action on $\mathcal{S}^{(AB)} \cong \mathbf{R}^{1,2}$ is the vector representation of $\mathbf{Spin}^+(\mathbf{1},\mathbf{2})$.

Given a spin frame $\{o^A, \iota^A\}$, i.e., a basis of $\mathcal{S}$ satisfying $\iota\cdot o = 1$, the elements

$$\Delta_1^{AB} := \frac{1}{\sqrt{2}}(o^Ao^B + \iota^A\iota^B) \qquad \Delta_2^{AB} := \frac{1}{\sqrt{2}}(o^Ao^B - \iota^A\iota^B)$$

$$\Delta_3^{AB} := \frac{1}{\sqrt{2}}(o^A\iota^B + \iota^Ao^B) = \sqrt{2}o^{(A}\iota^{B)}, \tag{18}$$

form a Ψ-ON basis for $\mathcal{S}^{(AB)}$. When the given spin frame is transformed by an element of $\mathbf{SL(2;R)} = \mathbf{Spin^+(1,2)}$, the corresponding Ψ-ON basis is transformed by the induced element of $\mathbf{SO^+(2,2)}$. Since the negative of the spin frame determines the same basis (18), there is a one-to-one correspondence between spin frames, up to sign, of $\mathcal{S}$ and the elements of a time-and-space orientation class of Ψ-ON bases of $\mathcal{S}^{(AB)}$.

With the usual definitions: $\Psi_0 := \Psi_{ABCD}o^Ao^Bo^Co^D$; $\Psi_1 = \Psi_{ABCD}o^Ao^Bo^C\iota^D$; $\Psi_2 = \Psi_{ABCD}o^Ao^B\iota^C\iota^D$; $\Psi_3 = \Psi_{ABCD}o^A\iota^B\iota^C\iota^D$; $\Psi_4 = \Psi_{ABCD}\iota^A\iota^B\iota^C\iota^D$; one has

$$\begin{aligned}\Psi_{ABCD} = {} & \Psi_0\iota_A\iota_B\iota_C\iota_D - 4\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)} \\ & - 4\Psi_3 o_{(A}o_Bo_C\iota_{D)} + \Psi_4 o_Ao_Bo_Co_D. \end{aligned} \tag{19}$$

One may now compute the matrix representation of $\Psi^{AB}{}_{CD}$ with respect to the basis (18):

$$\mathbf{\Psi} = \frac{1}{2}\begin{pmatrix} \Psi_0 + 2\Psi_2 + \Psi_4 & \Psi_0 - \Psi_4 & 2(\Psi_1 + \Psi_3) \\ -(\Psi_0 - \Psi_4) & -(\Psi_0 - 2\Psi_2 + \Psi_4) & -2(\Psi_1 - \Psi_3) \\ -2(\Psi_1 + \Psi_3) & -2(\Psi_1 - \Psi_3) & -4\Psi_2 \end{pmatrix}, \tag{20}$$

which is evidently trace free and self-adjoint in the sense of $\mathbf{R}^{1,2}$ as it should be (both properties can be directly verified abstractly for $\Psi^{AB}{}_{CD}$).

The elements $Z_i^{ab} := \Delta_i^{AB}\epsilon^{A'B'}$, $i = 1, 2, 3$, form a Ψ-ON basis of Λ_-^2 and

$$^{-}C^{ab}{}_{cd}\, Z_i^{cd} = 2\Psi^{AB}{}_{CD}\Delta_i^{CD}\epsilon^{A'B'}, \tag{21}$$

whence $^{-}\mathbf{C} = 2\mathbf{\Psi}$. Moreover, $\Psi^{AB}{}_{CD}\phi^{CD} = \lambda\phi^{AB}$ iff, with $F^{ab} = \phi^{AB}\epsilon^{A'B'}$, $^{-}C^{ab}{}_{cd}\, F^{cd} = 2\lambda F^{ab}$, i.e., the eigenvalue properties of $^{-}C^{ab}{}_{cd}$ are directly translatable into those of $\Psi^{AB}{}_{CD}$. In terms of the notation of [21], $^{-}C^{ab}{}_{cd}$ is W^-. The curvature classification given there is based on the possible Jordan canonical forms (JCF) of W^-. This classification therefore carries over immediately to $\Psi^{AB}{}_{CD}$. The goal is to relate this classification to that of (14).

The self-adjoint endomorphism $\Psi^{AB}{}_{CD}$ (which I shall also denote simply Ψ) has three (possibly complex, but in complex conjugate pairs) eigenvalues, λ_i, $i = 1, 2, 3$, from which one can form the invariants

$$0 = \mathrm{tr}(\Psi) \qquad I := \sum_{i=1}^{3}\lambda_i^2 = \mathrm{tr}(\Psi^2) \qquad J := \sum_{i=1}^{3}\lambda_i^3 = \mathrm{tr}(\Psi^3). \tag{22}$$

One may easily show that

$$\sum_{i<j}\lambda_i\lambda_j = -\frac{1}{2}I \qquad J = 3\lambda_1\lambda_2\lambda_3$$

$$\det(\Psi - \lambda 1) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - \frac{1}{2}I\lambda - \frac{J}{3}. \tag{23}$$

From (19) and (22) one finds, by squaring, respectively cubing, the first equation of (22),

$$I = 2\Psi_0\Psi_4 - 8\Psi_1\Psi_3 + 6(\Psi_2)^2 \qquad J = 3\det(\Psi) = 6\begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}. \tag{24}$$

From (23), the condition for eigenvalues with multiplicity greater than one is $I^3 = 6J^2$. If Ψ has an eigenvalue with algebraic multiplicity three (necessarily real), the trace free condition entails it must be zero. Hence, this condition is equivalent to $I = J = 0$.

5. Classification of the Weyl spinor

In this section I relate the classification of Ψ_{ABCD} provided by (14) to the classification induced by [21], treating the types of (14) one by one. First recall that the curvature types introduced in [21] are based on the possible JCFs of an endomorphism of $\mathbf{C}^3$:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Curvature type Ia occurs when W^- has JCF of the first kind and all three eigenvalues (not necessarily distinct) are real. Curvature type Ib is when there is a complex conjugate pair of eigenvalues for this first JCF. Curvature type II occurs when W^- has the second kind of JCF and curvature type III when it has the third kind of JCF.

Suppose ξ^A has components $\binom{1}{z}$ with respect to a spin frame $\{o^A, \iota^A\}$. Then

$$\begin{aligned} \Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D &= \Psi_0 + 4\Psi_1 z + 6\Psi_2 z^2 + 4\Psi_3 z^3 + \Psi_4 z^4 \\ &=: (\alpha_0 + \alpha_1 z)(\beta_0 + \beta_1 z)(\gamma_0 + \gamma_1 z)(\delta_0 + \delta_1 z) \end{aligned} \tag{25}$$

where the factorization is valid over $\mathbf{C}$ and defines the WPSs up to scale.

5.1. *Type* {1111}

With $\Psi^{ABCD} = \alpha^{(A}\beta^B\gamma^C\delta^{D)}$, $\alpha, \beta, \gamma, \delta \in \mathcal{S}$, one can scale the WPSs so that we may suppose

$$\alpha^A = 6\eta(o^A + \chi\iota^A) \qquad \beta^A = o^A \qquad \gamma^A = \lambda o^A + \iota^A \qquad \delta^A = \iota^A, \tag{26}$$

where $\{o^A, \iota^A\}$ is a spin frame, and $\eta, \chi, \lambda \in \mathbf{R}$ are nonzero, satisfying $\chi\lambda \neq 1$ so that $\gamma \cdot \alpha \neq 0$ (the factor 6 being for notational convenience). This form does, however, allow α^A to become coincident with any of the other WPSs by violating the constraints on the scalars. Indeed, in 5.1–5.3, the forms chosen to describe Ψ^{ABCD} are of course not uniquely determined, but are chosen to display degenerations that can occur as WPSs are made to coincide. Various other forms also serve this purpose.

From (25), one obtains with respect to the spin frame in (26),

$$\Psi_0 = 0 \qquad \Psi_1 = -\frac{3\eta\chi}{2} \qquad \Psi_2 = \eta(\lambda\chi + 1) \qquad \Psi_3 = -\frac{3\eta\lambda}{2} \qquad \Psi_4 = 0. \tag{27}$$

From (24) one finds,

$$I = 6\eta^2((\lambda\chi)^2 - \lambda\chi + 1) \qquad J = -3\eta^3(\lambda\chi + 1)(\lambda\chi - 2)(2\lambda\chi - 1), \tag{28}$$

whence from (23)

$$\lambda_1 = \eta(1 - 2\lambda\chi) \qquad \lambda_2 = \eta(1 + \lambda\chi) \qquad \lambda_3 = \eta(\lambda\chi - 2). \tag{29}$$

Thus, Ψ has three real eigenvalues which are distinct when Ψ^{ABCD} is nonzero and type {1111}, i.e., no pair of α, β, γ and δ in (26) coincide in direction. Thus, type {1111} is of curvature type Ia, i.e., Ψ has diagonal real JCF. Since Ψ is self-adjoint, there is another spin frame $\{o^A, \iota^A\}$ for which the associated basis (18) is an eigenbasis, so Ψ is diagonalized: $\boldsymbol{\Psi} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Setting this diagonal form equal to (20) yields

$$\Psi_0 = \frac{\lambda_1 - \lambda_2}{2} \qquad \Psi_1 = 0 \qquad \Psi_2 = -\frac{\lambda_3}{2} \qquad \Psi_3 = 0 \qquad \Psi_4 = \frac{\lambda_1 - \lambda_2}{2} \tag{30}$$

resulting in a canonical form

$$\Psi^{ABCD} = \frac{\lambda_1 - \lambda_2}{2}(\iota^A\iota^B\iota^C\iota^D + o^A o^B o^C o^D) - 3\lambda_3 o^{(A} o^B \iota^C \iota^{D)}. \tag{31}$$

To emphasize the difference between the spin frames yielding (27) and (30), the former determines WPSs of Ψ^{ABCD} while the latter determines the eigenvectors of Ψ through (18).

5.2. *Type* $\{1\overline{1}11\}$

With $\Psi^{ABCD} = \alpha^{(A}\overline{\alpha}^B\gamma^C\delta^{D)}$, $\alpha \in \mathbf{C}\mathcal{S}$, $\gamma, \delta \in \mathcal{S}$, take $\iota^A := \delta^A$. Using the freedom to scale α by $e^{i\theta}$, one can arrange that $\Im(\alpha^A) \propto \iota^A$ without changing the given symmetric product. By (9), $\Im(\alpha) \cdot \Re(\alpha) \neq 0$, so one can choose a spin frame $\{o^A, \iota^A\}$ so that

$$\alpha^A = \sqrt{6}\eta(o^A + i\chi\iota^A) \qquad \gamma^A = \lambda o^A + \iota^A \qquad \delta^A = \iota^A, \tag{32}$$

where $\eta, \chi, \lambda \in \mathbf{R}$ are nonzero. Relaxing these conditions on the scalars results in coincidences of the WPSs.

From (25), one obtains with respect to the spin frame in (32)

$$\Psi_0 = 6\eta^2\chi^2 \qquad \Psi_1 = -\frac{3\lambda\eta^2\chi^2}{2} \qquad \Psi_2 = \eta^2 \qquad \Psi_3 = -\frac{3\lambda\eta^2}{2} \qquad \psi_4 = 0. \tag{33}$$

From (24)

$$I = 6\eta^4(1 - 3\lambda^2\chi^2) \qquad J = -6\eta^6(1 + 9\lambda^2\chi^2) \tag{34}$$

whence from (23)

$$\lambda_1 = \eta^2(1 + 3i\lambda\chi) \qquad \lambda_2 = \eta^2(1 - 3i\chi) \qquad \lambda_3 = -2\eta^2. \tag{35}$$

Thus, Ψ has three distinct eigenvalues, one real and a complex conjugate pair, whence Ψ is of curvature type Ib. In this case, there is another spin frame $\{o^A, \iota^A\}$ which determines a Ψ-ON frame (18) wrt which Ψ has matrix representation

$$\begin{pmatrix} \Re(\lambda_1) & \Im(\lambda_1) & 0 \\ -\Im(\lambda_1) & \Re(\lambda_1) & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

see [21]. Equating this expression to (20) yields, noting $\lambda_2 = \overline{\lambda_1}$,

$$\Psi_0 = \frac{\lambda_1 - \lambda_2}{2i} \qquad \Psi_1 = 0 \qquad \Psi_2 = -\frac{\lambda_3}{2} \qquad \Psi_3 = 0 \qquad \Psi_4 = -\frac{\lambda_1 - \lambda_2}{2i}, \tag{36}$$

resulting in a canonical form

$$\Psi^{ABCD} = \frac{\lambda_1 - \lambda_2}{2i}(\iota^A\iota^B\iota^C\iota^D - o^A o^B o^C o^D) - 3\lambda_3 o^{(A}o^B\iota^C\iota^{D)}. \tag{37}$$

Again, the spin frame of (33) determines the WPSs while the spin frame of (36) determines the eigenvectors of Ψ. In particular, Δ_3^{AB} is an eigenvector for λ_3 while $\Delta_1^{AB} + i\Delta_2^{AB}$ is an eigenvector for λ_1.

5.3. *Type* $\{1\bar{1}1\bar{1}\}$

With $\Psi^{ABCD} = \alpha^{(A}\overline{\alpha}^B\beta^C\overline{\beta}^{D)}$, $\alpha, \beta \in \mathbf{C}\mathcal{S}$, scale β^A by $e^{i\theta}$ so that $\Im(\beta^A) \propto \Im(\alpha^A)$. Then one can choose a spin frame $\{o^A, \iota^A\}$ so that

$$\alpha^A = o^A + i\chi\iota^A \qquad \beta^A = \sqrt{6}(\lambda o^A + \eta\iota^A + i\rho\iota^A), \tag{38}$$

where χ, λ, η and $\rho \in \mathbf{R}$, χ and ρ nonzero, and η and $\rho \mp \lambda\chi$ are not both zero (so $\beta \cdot \alpha$, respectively $\beta \cdot \overline{\alpha}$, are nonzero).

From (25), one obtains with respect to the spin frame in (38),

$$\Psi_0 = 6\chi^2(\eta^2 + \rho^2) \qquad \Psi_1 = -\frac{3\eta\lambda\chi^2}{2} \qquad \Psi_2 = \eta^2 + \rho^2 + \chi^2\lambda^2 =: \omega$$
$$\Psi_3 = -\frac{3\eta\lambda}{2} \qquad \Psi_4 = 6\lambda^2. \tag{39}$$

From (24)

$$I = 6(\omega^2 + 12\chi^2\lambda^2\rho^2) \qquad J = 6\omega(36\chi^2\lambda^2\rho^2 - \omega^2), \tag{40}$$

whence from (23)

$$\lambda_1 = \omega + 6\chi\lambda\rho \qquad \lambda_2 = \omega - 6\chi\lambda\rho \qquad \lambda_3 = -2\omega. \tag{41}$$

Granted the constraints on the scalars χ, λ, η and ρ, these three eigenvalues are real and distinct. Thus, Ψ is of curvature type Ia. The canonical form is therefore exactly as for type $\{1111\}$, i.e., (30) and (31) but with (41) defining the eigenvalues.

5.4. *Type* $\{211\}$

Type $\{211\}$ is a degenerate form of $\{1111\}$; for example put $\chi = 0$ to get $\beta \propto \alpha$; $\lambda = 0$ to get $\gamma \propto \delta$; other coincidences are also possible. It is also a degenerate form of $\{1\bar{1}11\}$ on putting $\chi = 0$. χ and λ play symmetrical roles in (27). Restricting attention to $\chi = 0$ in each of $\{1111\}$ and $\{1\bar{1}11\}$, one obtains essentially the same result (the only difference being that $\alpha = 6\beta$ from $\{1111\}$ while $\beta = \alpha$ from $\{1\bar{1}11\}$). Working with the form obtained from $\{1111\}$ with $\chi = 0$,

$$\Psi_0 = \Psi_1 = \Psi_4 = 0 \qquad \Psi_2 = \eta \qquad \Psi_3 = -3\eta\lambda/2, \tag{42}$$

i.e.,

$$\Psi^{ABCD} = 6\eta(o^{(A}o^B\iota^C\iota^{D)} + \lambda o^{(A}o^B o^C\iota^{D)}). \tag{43}$$

From (28), $I = 6\eta^2$ and $J = -6\eta^3$ so $I^3 = 6J^2$ indicating eigenvalues with multiplicity greater than one. Indeed, from (29)

$$\lambda_1 = \lambda_2 = \eta \qquad \lambda_3 = -2\eta. \tag{44}$$

Indeed, substituting (42) into (20) yields a matrix with eigenvalues: η of algebraic multiplicity two and geometric multiplicity one with eigenvector $\Delta_1 + \Delta_2$; -2η of algebraic and geometric multiplicities one with eigenvector $\Delta_1 + \Delta_2 + 2\Delta_3/\lambda$; where Δ_1, Δ_2, Δ_3 are as in (18) for the spin frame yielding (42). These properties determine a JCF characterizing curvature type II.

Note that since this JCF is different to that underlying the form (31), setting $\lambda_1 = \lambda_2$ in (31) does *not* yield a valid form for type {211}. If one rescales the spin frame underlying (43): $o^A \mapsto \mu^{-1}o^A$, $\iota^A \mapsto \mu\iota^A$, for some nonzero real μ, the factor λ in (43) can be absorbed to yield a canonical form for {211}, viz., put $\lambda = 1$ in (43).

5.5. *Type* $\{1\bar{1}2\}$

Type $\{1\bar{1}2\}$ can be obtained from either $\{1\bar{1}11\}$ ($\lambda = 0$) or $\{1\bar{1}1\bar{1}\}$ ($\rho = 0$). The latter yields a more general expression, but it can always be recast into the form obtained from $\{1\bar{1}11\}$ by suitable scaling of the WPSs. Putting $\lambda = 0$ in (33) yields

$$\Psi_0 = 6\eta^2\chi^2 \qquad \Psi_2 = \eta^2 \qquad \Psi_1 = \Psi_3 = \Psi_4 = 0, \tag{45}$$

so

$$\Psi^{ABCD} = 6\eta^2(\chi^2\iota^A\iota^B\iota^C\iota^D + o^{(A}o^B\iota^C\iota^{D)}). \tag{46}$$

From (34), $I = 6\eta^4$ and $J = -6\eta^6$ so $I^3 = 6J^2$. Putting $\lambda = 0$ in (35) gives

$$\lambda_1 = \eta^2 = \lambda_2 \qquad \lambda_3 = -2\eta^2. \tag{47}$$

Indeed, substituting (45) into (20) yields a matrix whose eigenvalues are: η^2 of algebraic multiplicity two and geometric multiplicity one with eigenvector $\Delta_1 - \Delta_2$; $-2\eta^2$ of algebraic and geometric multiplicities one with eigenvector Δ_3. Hence, this information also determines a JCF characterizing curvature type II.

The form (37), with $\lambda_1 = \lambda_2$ is not valid for $\{1\bar{1}2\}$, being based on a distinct JCF. Rescaling the spin frame in (46) allows one to absorb the χ^2 factor yielding a canonical form for $\{1\bar{1}2\}$, viz., put $\chi^2 = 1$ in (46).

5.6. *Type* {22}

This type can be obtained from either {211} or $\{1\bar{1}2\}$. Considering the former possibility, for example, putting $\chi = \lambda = 0$ in (27) results in $\Psi_2 = \eta$ as the only nonzero dyad component of Ψ_{ABCD}, whence $I = 6\eta^2$, $J = -6\eta^3$, and $I^3 = 6J^2$. From (29) or (44)

$$\lambda_1 = \lambda_2 = \eta \qquad \lambda_3 = -2\eta. \tag{48}$$

Indeed, (20) yields a diagonal matrix $\mathbf{\Psi} = \mathrm{diag}(\eta, \eta, -2\eta)$ indicating curvature type Ia.

Of course, $\Psi^{ABCD} = 6\eta o^{(A}o^B\iota^C\iota^{D)}$ is a canonical form for type {2, 2}. Indeed, since the curvature type is Ia, the canonical form (31) must be valid with $\lambda_1 = \lambda_2$ and $\lambda_3 = -2\eta$. The basis (18) associated with the spin frame giving this canonical form is, by construction, an eigenbasis, with Δ_i an eigenvector of λ_i, $i = 1, 2, 3$.

5.7. *Type* $\{2\bar{2}\}$

For type $\{2\bar{2}\}$, put $\rho = \chi$, $\eta = 0$ and $\lambda = 1$ in (38) and (39) to yield

$$\Psi_0 = 6\chi^4 \qquad \Psi_2 = 2\chi^2 \qquad \Psi_1 = \Psi_3 = 0 \qquad \Psi_4 = 6. \tag{49}$$

One finds that $I^3 = 6J^2$ and from (41)

$$\lambda_1 = 8\chi^2 \qquad \lambda_2 = \lambda_3 = -4\chi^2. \tag{50}$$

Indeed substituting (49) into (20) yields a matrix whose eigenvalues are: $8\chi^2$ of algebraic and geometric multiplicities one with eigenvector $(1+\chi^2)\Delta_1 + (1-\chi^2)\Delta_2$; $-4\chi^2$ with algebraic and geometric multiplicities two with eigenvectors Δ_3 and $(1-\chi^2)\Delta_1 + (1+\chi^2)\Delta_2$; Δ_1, $\Delta_2\Delta_3$ as in (18) for the spin frame determining (49). Thus, Ψ is of curvature type Ia.

It follows that the canonical form (31) must be valid for type $\{2\bar{2}\}$ upon setting $\lambda_2 = \lambda_3$, with the result, using the values (50),

$$\Psi^{ABCD} = 6\chi^2(o^A o^B o^C o^D + \iota^A \iota^B \iota^C \iota^D + 2o^{(A} o^B \iota^C \iota^{D)}). \tag{51}$$

It is straightforward to confirm that the basis (18) for this spin frame is indeed an eigenbasis, with Δ_i an eigenvector of λ_i.

5.8. *Type* $\{31\}$

Type $\{31\}$ can be obtained from type $\{1111\}$. It is convenient to take a slightly different form than (26) to describe this degeneration; namely

$$\alpha^A = 6\eta(o^A + \chi\iota^A) \qquad \beta^A = o^A \qquad \gamma^A = o^A + \mu\iota^A \qquad \delta^A = \iota^A, \tag{52}$$

which, in effect is obtained from (26) by putting $\mu = \lambda^{-1}$. The expressions corresponding to (27)–(29) are easily obtained, either directly, or by noting that one need only insert a factor of μ into each of (27), whence μ^2 into I and μ^3 into J in (28) and then a factor of μ into each of (29), with the understanding that $\mu\lambda = 1$. Degeneration to type $\{31\}$ is then achieved by setting $\chi = \mu = 0$. It follows: that $\Psi_3 = -3\eta/2$ is the only nonzero dyad component of the type $\{31\}$ form obtained so

$$\Psi^{ABCD} = 6\eta o^{(A} o^B o^C \iota^{D)}; \tag{53}$$

that $I = J = 0$; and that zero is the single eigenvalue. Indeed, computing the matrix (20) for Ψ with these dyad components, one easily finds that zero is the only eigenvalue, but has geometric multiplicity only one, with eigenvector $\Delta_1 + \Delta_2$ in fact. It follows that the curvature type is III.

Note that the freedom to rescale a spin frame permits one to absorb the scalar coefficient in (53) to give a canonical form $o^{(A} o^B o^C \iota^{D)}$ for type $\{31\}$.

5.9. *Type* $\{4\}$

Type $\{4\}$ obviously possesses the canonical form

$$\Psi^{ABCD} = \pm o^A o^B o^C o^D. \tag{54}$$

Computing (20) for this form ($\Psi_4 = \pm 1$ is the only nonzero dyad component, so $I = J = 0$), one obtains a matrix whose only eigenvalue is zero of geometric multiplicity two, with eigenvectors

$\Delta_1 + \Delta_2$ and Δ_3. It follows that the curvature type is II (indeed $\mathbf{\Psi}$ is already in the form characteristic of curvature type II with respect to a Ψ-ON basis of $\mathbf{R}^{1,2}$ given in [21]).

Type $\{4\}$ can arise by degeneration directly from types $\{31\}$, $\{22\}$ and $\{2\bar{2}\}$. One must employ noncanonical forms for $\{31\}$, $\{22\}$ and $\{2\bar{2}\}$ for an explicit description, however. For example, type $\{4\}$ can be obtained explicitly from type $\{2\bar{2}\}$ by putting $\chi = 0$ in (49) and one sees two distinct eigenvectors become coincident. On the other hand, putting $\rho = 0$ in (38) passes from $\{1\bar{1}1\bar{1}\}$ to $\{1\bar{1}2\}$, then setting $\chi = 0$ yields a noncanonical form of $\{22\}$, which with $\eta = 0$ reduces to type $\{4\}$.

The particular noncanonical forms I have employed are certainly not exhaustive for describing degenerations; in particular, they cannot describe the degeneration through $\{31\}$ to $\{4\}$. But other forms could of course be constructed to do so if desired.

The classification of the Weyl spinor is now complete and summarized in the following diagram, where m denotes the algebraic and M the geometric multiplicity of an eigenvalue:

#Eigenvalues Multiplicities										
3 $m = M = 1$ each	$\{1111\}$Ia				$\{1\bar{1}11\}$Ib				$\{1\bar{1}1\bar{1}\}$Ia	Algebraically General
		$\searrow$		$\swarrow$		$\searrow$		$\swarrow$		
2 $m = 2\ M = 1$ $m = M = 1$			$\{211\}$II				$\{1\bar{1}2\}$II		$\downarrow$	$I^3 = 6J^2 \neq 0$
				$\searrow$		$\swarrow$				
2 $m = M = 2$ $m = M = 1$					$\{22\}$Ia				$\{2\bar{2}\}$Ia	$I^3 = 6J^2 \neq 0$
			$\downarrow$			$\searrow$		$\swarrow$		
1 $m = 3\ M = 2$							$\{4\}$II			$I = J = 0$
						$\nearrow$				
1 $m = 3\ M = 1$			$\{31\}$III		$\longrightarrow$					$I = J = 0$

In [21], for each curvature type I gave a matrix representation with respect to a Ψ-ON basis of $\mathbf{R}^{1,2} \cong \Lambda^2_- \cong \mathcal{S}^{(AB)}$ as an alternative form to the corresponding JCF. I close by asking to what extent those forms are canonical. For curvature type Ia, the JCF is real diagonal; equating to (20) to determine a form for Ψ^{ABCD} is exactly what was done in 5.1 and 5.3 for $\{1111\}$ and $\{1\bar{1}1\bar{1}\}$ respectively. For curvature type Ib, the JCF is diagonal over $\mathbf{C}$. Transforming to a real Ψ-ON basis and equating to (20) is exactly what was done in 5.2 for $\{1\bar{1}11\}$. Thus, the characteristic forms given in [21] for curvature types Ia and Ib determine the canonical forms given for Ψ^{ABCD} in 5.1–5.3. The algebraically special curvature type Ia forms, viz., $\{22\}$ and $\{2\bar{2}\}$, are obtained from (31) by imposing the appropriate conditions on the eigenvalues.

For curvature type II, there are two forms given in [21]:

$$\begin{pmatrix} \mu + 1/2 & -1/2 & 0 \\ 1/2 & \mu - 1/2 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \mu - 1/2 & 1/2 & 0 \\ -1/2 & \mu + 1/2 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Equating the first to (20) yields $\Psi^{ABCD} = 6\mu o^{(A} o^B \iota^C \iota^{D)} + o^A o^B o^C o^D$. Under a suitable spin transformation, this form is equivalent to (46) (with $\chi = 1$), i.e., type $\{1\bar{1}2\}$. Equating the second displayed matrix to (20) yields $\Psi^{ABCD} = 6\mu o^{(A} o^B \iota^C \iota^{D)} - o^A o^B o^C o^D$, which, under a suitable spin transformation, is equivalent to (43) (with $\lambda = 1$), i.e., type {211}. Thus, the instances of (20) dictated by (46) (with $\chi = 1$) and (43) (with $\lambda = 1$) are, at least from a spinorial point of view, more canonical than the forms given for curvature type II in [21]. The canonical forms (54) of the algebraically special form of curvature type II, {4}, are obviously the special cases of the forms derived from [21] with $0 = \mu = -\lambda/2$.

The matrix representation given for curvature type III in [21] determines, via (20), an expression for Ψ^{ABCD} of the form (53). Thus, it is not quite canonical, but only a scaling of the spin frame is required to achieve such.

The geometry and symmetries of the Weyl curvature spinor and other applications will be treated elsewhere.

Acknowledgments

This work was done some 10 years ago. I thank Yasuo Matsushita for motivating me to write it up and Roger Penrose for his abiding influence. I thank an anonymous referee for remarks that improved the manuscript.

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